Algebra Rules and Equations

Linear Equations

1. General form of a linear equation: $Ax + By + C = 0$
2. Slope formula: $m = \frac{change\ in\ y}{change\ in\ x} = \frac{y_2 - y_1}{x_2 - x_1}$
3. Slope intercept form: $y = mx + b$ Where m is the slope and b is the y-intercept
4. Point-slope form (used to find the equation of a line): $y - y_1 = m(x - x_1)$
5. Horizontal line parallel to the x-axis: $y = b$
6. Vertical line parallel to the y-axis: $x = a$

Distance and Midpoint Formulas

1. Distance d from the points (x_1, y_1) to (x_2, y_2): $d = \sqrt{(x_2 - x_1)^2 + (y_2 - y_1)^2}$
2. Midpoint of a line segment with endpoints (x_1, y_1) and (x_2, y_2):
   $\left(\frac{x_1 + x_2}{2}, \frac{y_1 + y_2}{2}\right)$

Quadratic Formula

The solutions of $ax^2 + bx + c = 0$ with $a \neq 0$ are given by:

$$x = \frac{-b \pm \sqrt{b^2 - 4ac}}{2a}$$

The expression $b^2 - 4ac$ in the quadratic formula is the discriminant. The following rules apply to the discriminant:

If $b^2 - 4ac = 0$ there is one real number solution
If $b^2 - 4ac > 0$ there are two different real number solutions
If $b^2 - 4ac < 0$ there are two different imaginary number solutions

Completing the Square

To solve a quadratic equation by completing the square:

1. Isolate the terms with variables on one side of the equation and arrange them in descending order.
2. If the a coefficient is not 1 then divide every term on both sides of the equation by a
3. Subtract the constant c (or add the opposite of c) from both sides of the equation

4. Divide the *b* coefficient by 2; square the result and add to both sides of the equation (this will transform the left side of the equation into a perfect square trinomial)
5. Factor and express one side of the equation as the square of a binomial (in the form $(x+a)^2$).
6. Using the square root principal and the quotient rule for radicals take the square root of both sides of the equation remembering the $\pm$ for the square root.
7. Solve for the variable (typically *x*).
8. When used for solving equation of a circle collect the terms involving *x* and those involving *y* and complete the square for each

Example:

$x^2-6x-10=0$	
$x^2-6x=10$	Add 10 to both sides of the equation
$x^2-6x+9=10+9$	Take half the coefficient of the *x*-term (6*x*), square it (3^2) and add to both sides
$x^2-6x+9=19$	
$(x-3)^2=19$	Factor x^2-6x+9
$x-3=\pm\sqrt{19}$	Take the square root of both sides of the equation (principle of square roots)
$x=3\pm\sqrt{19}$	Add 3 to both sides of the equation

Completing the square for conic sections with two terms:

1. Group the *x* terms together and the *y* terms together
2. Use the steps for completing the square above and complete the square twice, once for the *x* group of terms and once for the *y* group of terms.

Example:

Find the center and radius for the following equation of a circle:

$x^2+y^2-16x+14y+32=0$	
$x^2+y^2-16x+14y=-32$	Isolate the terms with variables on one side of the equation
$x^2-16x+y^2+14y=-32$	Group the *x* and *y* terms together
$(x^2-16x+64)+(y^2+14y+49)=-32+64+49$	
	Take half the coefficient of the first degree term and add the squares to both sides of the equation $\left[\frac{1}{2}(-16)\right]^2=(-8)^2=64$ and $\left(\frac{1}{2}\cdot 14\right)^2=7^2=49$
$(x-8)^2+(y+7)^2=81$	Factor and express as the square of the binomial

$(x-8)^2 + [y-(-7)]^2 = 9^2$ Write in standard form where the center is (8, -7) and the radius is 9

Conic Sections

General form of the conic equation: $Ax^2 + Bx + Cy^2 + Dy + E = 0$

To determine what conic section the equation represents:

If $A = 0$ or $C = 0$ (the x^2 or y^2 term is missing) the equation is a parabola.
If $A = C$ the equation is a circle.
If $A \neq C$ but they both have the same sign, then the equation is an ellipse.
If $A \neq C$ and they have opposite signs, then the equation is a hyperbola.

Parabola

Equation of a parabola with vertex at the origin (0, 0) and directrix $y = -p$ is:

$x^2 = 4py$ Where p is the directed distance from the vertex to either the focus or the directrix, the focus is $(0, p)$ and the y-axis is the axis of symmetry.

Equation of a parabola with vertex at the origin (0, 0) and directrix $x = -p$ is:

$y^2 = 4px$ Where p is the directed distance from the vertex to either the focus or the directrix, the focus is $(p, 0)$ and the x-axis is the axis of symmetry.

Equation of a parabola with vertex at the coordinate (h, k) and vertical axis symmetry:

$$(x-h)^2 = 4p(y-k)$$

Vertex: (h, k)
Focus: $(h, k + p)$
Directrix: $y = k - p$
Axis of symmetry: $x = h$

Equation of a parabola with vertex at the coordinate (h, k) and horizontal axis symmetry:

$$(y-k)^2 = 4p(x-h)$$

Vertex: (h, k)
Focus: $(h + p, k)$
Directrix: $x = h - p$
Axis of symmetry: $y = k$

Example:

Determine the coordinates of the vertex, focus, directrix, and axis of symmetry of the parabola whose equation is:

$y^2 - 6y + 8x + 17 = 0$	
$y^2 - 6y = -8x - 17$	Subtract 8*x* and 17 from both sides of the equation
$y^2 - 6y + 9 = -8x - 17 + 9$	Complete the square on the left side of the equation
$(y-3)^2 = -8x - 8$	Factor left side and simplify right side of equation
$(y-3)^2 = -8(x+1)$	Factor right side of equation

The equation is now in standard form $(y-k)^2 = 4p(x-h)$ where h = -1, k = 3, and 4p = -8 so p = -2. The vertex is (h, k) = (-1, 3), the focus is at (h + p, k) = (-1 + -2, 3) = (-3, 3), and the directrix is at $x = h – p$ = -1 – (-2) = 1. Since $p < 0$ or negative, the parabola opens to the left.

Circle

Equation of a circle: $(x-h)^2 + (y-k)^2 = r^2$
Equation of a circle with center at the origin: $x^2 + y^2 = r^2$

Ellipse

Ellipse with center at the origin (0, 0) major axis horizontal:

$$\frac{x^2}{a^2} + \frac{y^2}{b^2} = 1, a > b > 0$$

Where a represents the distance between the center and a vertex along the major axis and b represents the distance between the center and an endpoint on the minor axis.
Vertices: (-a, 0), (a, 0)
y-intercepts: (0, -b), (0, b)
Foci: (-c, 0), (c, 0), where $c^2 = a^2 - b^2$

Ellipse with center at the origin (0, 0) major axis vertical:

$$\frac{x^2}{b^2} + \frac{y^2}{a^2} = 1, a > b > 0$$

Vertices: (0, -a), (0, a)
x-intercepts: (-b, 0), (b, 0)
Foci: (0, -c), (0, c), where $c^2 = a^2 - b^2$

Equation of an ellipse with center at (h, k) major axis horizontal:

$$\frac{(x-h)^2}{a^2}+\frac{(y-k)^2}{b^2}=1, a>b>0$$

Vertices: $(h - a, k)$, $(h + a, k)$
Length of minor axis: $2b$
Foci: $(h - c, k)$, $(h + c, k)$, where $c^2 = a^2 - b^2$

Equation of an ellipse with center at (h, k) major axis vertical:

$$\frac{(x-h)^2}{b^2}+\frac{(y-k)^2}{a^2}=1, a>b>0$$

Vertices: $(h, k - a)$, $(h, k + a)$
Length of minor axis: $2b$
Foci: $(h, k - c)$, $(h, k + c)$, where $c^2 = a^2 - b^2$

Find the vertex, the foci, and the length of the minor axis for the following equation:

Example:

Find the vertices, the foci, and the y-intercept for the following equation:

$$9x^2+4y^2=36$$

$$\frac{1}{36}(9x^2+4y^2)=36\cdot\frac{1}{36}$$ Multiply both sides by $\frac{1}{36}$ to get a 1 on the right side

$$\frac{9x^2}{36}+\frac{4y^2}{36}=1$$

$$\frac{x^2}{4}+\frac{y^2}{9}=1$$

$$\frac{x^2}{2^2}+\frac{y^2}{3^2}=1$$ Write in standard form

The constant beneath the *x*-expression is less than the constant below the *y*-expression so the major axis of the ellipse is vertical. If the constant under the *x*-expression is greater than the constant beneath the *y*-expression the major axis is horizontal. When the major axis is vertical the b constant is the denominator of the *x*-expression. When the major axis is horizontal the a constant is the denominator of the *x*-expression. The vertices are $(0, -a)$, $(0, a)$ or $(0, -3)$, $(0, 3)$. Since $b = 2$ the *x*-intercepts are $(-2, 0)$ and $(2, 0)$.

To find the foci (0, -c), (0, c), solve $c^2 = a^2 - b^2$ for c as follows:

$$c^2 = a^2 - b^2$$
$$c^2 = 3^2 - 2^2$$
$$c^2 = 9 - 4$$
$$c^2 = 5$$
$$c = \sqrt{5} \qquad \text{Take the square root of both sides of the equation}$$
$$c = \sqrt{5}$$

The foci are $\left(0, -\sqrt{5}\right)$ and $\left(0, \sqrt{5}\right)$.

Hyperbola

Equation of a hyperbola with center at the origin (0, 0) and transverse axis horizontal:

$$\frac{x^2}{a^2} - \frac{y^2}{b^2} = 1$$

Vertices: (-a, 0), (a, 0)

Asymptotes: $y = -\frac{b}{a}x,\ y = \frac{b}{a}x$

Foci: (-c, 0), (c, 0), where $c^2 = a^2 + b^2$

Equation of a hyperbola with center at the origin (0, 0) and transverse axis vertical:

$$\frac{y^2}{a^2} - \frac{x^2}{b^2} = 1$$

Vertices: (0, -a), (0, a)

Asymptotes: $y = -\frac{a}{b}x,\ y = \frac{a}{b}x$

Foci: (0, -c), (0, c), where $c^2 = a^2 + b^2$

Equation of a hyperbola with center at (h, k) and transverse axis horizontal:

$$\frac{(x-h)^2}{a^2} - \frac{(y-k)^2}{b^2} = 1$$

Vertices: (h - a, k), (h + a, k)

Asymptotes: $y - k = \frac{b}{a}(x - h)$

$y - k = -\frac{b}{a}(x - h)$

Foci: (h – c, k), (h + c, k), where $c^2 = a^2 + b^2$

Equation of a hyperbola with center at (h, k) and transverse axis vertical:

$$\frac{(y-k)^2}{a^2}-\frac{(x-h)^2}{b^2}=1$$

Vertices: (h, k - a), (h, k + a)

Asymptotes: $y-k=\frac{a}{b}(x-h)$

$$y-k=-\frac{a}{b}(x-h)$$

Foci: (h, k – c), (h, k + c), where $c^2=a^2+b^2$

Example:

Find the vertices, the foci and the asymptotes for the hyperbola given by:

$$9x^2-16y^2=144$$

$$\frac{1}{144}(9x^2-16y^2)=144\cdot\frac{1}{144}$$ Multiply both sides by $\frac{1}{144}$ to get 1 on the right side

$$\frac{x^2}{16}-\frac{y^2}{9}=1$$

$$\frac{x^2}{4^2}-\frac{y^2}{3^2}=1$$ Write in standard form

Since this equation is a hyperbola with center at the origin and transverse axis horizontal the vertices are (-a, 0), (a, 0), or (-4, 0), (4, 0). To find the foci solve $c^2=a^2+b^2$ for c as follows:

$$c^2=a^2+b^2$$
$$c^2=4^2+3^2$$
$$c^2=16+9$$
$$c^2=25$$
$$c=\sqrt{25}$$ Take the square root of both sides of the equation
$$c=5$$

The foci are (-c, 0), (c, 0) or (-5, 0), (5, 0).
The asymptotes are as follows:

$$y=-\frac{b}{a}x,\ y=\frac{b}{a}x$$

$$y=-\frac{3}{4}x,\ y=\frac{3}{4}x$$

Special Factors

Examples

$x^2 - a^2 = (x-a)(x+a)$ — $x^2 - 9 = (x-3)(x+3)$

$x^3 - a^3 = (x-a)(x^2+ax+a^2)$ — $x^3 - 8 = (x-2)(x^2+2x+4)$

$x^3 + a^3 = (x+a)(x^2-ax+a^2)$ — $x^3 + a^3 = (x+a)(x^2-ax+a^2)$

$x^4 - a^4 = (x-a)(x+a)(x^2+a^2)$ — $x^4 - 4 = \left(x-\sqrt{2}\right)\left(x+\sqrt{2}\right)\left(x^2+2\right)$

$x^4 + a^4 = (x^2+\sqrt{2ax}+a^2)(x^2-\sqrt{2ax}+a^2)$ — $x^4 - 4 = (x^2+2x+2)(x^2-2x+2)$

Binomial Expansions

Examples

$(a+b)^0 = 1$

$(a+b)^1 = a+b$

$(a+b)^2 = a^2+2ab+b^2$ — $(x+3)^2 = x^2+6x+9$

$(a+b)^3 = a^3+3a^2b+3ab^2+b^3$ — $(x+2)^3 = x^3+6x^2+12x+8$

$(a+b)^4 = a^4+4a^3b+6a^2b^2+4ab^3+b^4$ — $(x+2)^4 = x^4+8x^3+24x^2+32x+16$

$(a+b)^5 = a^5+5a^4b+10a^3b^2+10a^2b^3+5ab^4+b^5$

Binomial Theorem Using Pascal's Triangle

For any binomial *a* + *b* and any natural number *n*,

$$(a+b)^n = c_0a^nb^0 + c_1a^{n-1}b^1 + c_2a^{n-2}b^2 + ... + c_{n-1}a^1b^{n-1} + c_na^0b^n$$

Where the numbers $c_0, c_1, c_2, ..., c_{n-1}, c_n$ are from the (*n* + 1) row of Pascal's triangle

Factoring by Grouping

Given ax^2+bx+c to factor, where *a*, *b*, and *c* are real numbers and $a \neq 1$:

1. Factor out the largest common factor
2. Find two integers whose sum is *b* and whose product is *ac*.
3. Split the middle term by writing it as a sum using the factors found in step 2.
4. Factor by grouping.

Example:

Factor $12x^3 + 10x^2 - 8x$

Factor out the largest common factor 2x:
$12x^3 + 10x^2 - 8x = 2x(6x^2 + 5x - 4)$

Multiply the leading coefficient 6, times the constant -4, $6 \cdot (-4) = -24$.

Find the factors of *ac* = -24 so that the sum of the factors is *b* = 5. In this case the factors are $-3 \cdot 8 = -24; -3 + 8 = 5$.

Split the middle term using the factors found above $5x = -3x + 8x$

Factor by grouping:

$6x^2 + 5x - 4 = 6x^2 - 3x + 8x - 4$	
$= 3x(2x-1) + 4(2x-1)$	Factor 3x out of $6x^2 - 3x$ and 4 out of $8x - 4$
$= (3x+4)(2x-1)$	Factor by grouping
$12x^3 + 10x^2 - 8x = 2x(3x+4)(2x-1)$	Include the common factor to get the complete factorization of the original trinomial