Complex Numbers

1. Imaginary unit: $i=\sqrt{-1}$, $\sqrt{-a}=i\sqrt{a}, a\geq 0$
2. Powers of *i*. This pattern repeats every fourth time.

$$i^0=1$$
$$i^1=\sqrt{-1}$$
$$i^2=-1$$
$$i^3=-i$$
$$i^4=1$$

3. Definition of a complex number: $a+bi$ where a and b are real numbers. If $b\neq 0, a+bi$ is called an imaginary number.
4. Complex conjugate: $\overline{(a+bi)}=a-bi$
5. If $a+bi$ and $c+di$ are two complex numbers written in standard form, their sum and difference are defined as follows:

Sum: $(a+bi)+(c+di)=(a+c)+(b+d)i$
Example: $(3+4i)+(2+6i)=(3+2)+(4i+6i)=5+10i$

Difference: $(a+bi)-(c+di)=(a-c)+(b-d)i$
Example: $(6+3i)-(10+5i)=(6-10)+(3i-5i)=-4-2i$

6. Multiplication of complex numbers:

$$(a+bi)(c+di)=ac+(ad+bc)i+bdi^2=(ac-bd)+(ad+bc)i$$

Example:

$\left(3+2i\right)(3-2i)=9-6i+6i-4i^2$ Product of binomials
$=9-4(-1)$ $i^2=-1$
$=9+4$ Simplify
$=13$

7. Division of complex numbers:

$$\frac{a+bi}{c+di}=\frac{a+bi}{c+di}\cdot\frac{c-di}{c-di}=\frac{(ac+bd)+(bc-ad)i}{c^2+d^2}$$

Example:

$$\frac{2+3i}{4-2i}$$

$$\frac{2+3i}{4-2i}\cdot\frac{4+2i}{4+2i}$$ Multiply by the conjugate of the denominator

$$=\frac{8+4i+12i+6i^2}{16-4i^2}$$ $i^2=-1$

$$=\frac{8-6+16i}{16+4}=\frac{2+16i}{20}=\frac{1}{10}+\frac{4}{5}i$$ Simplify and write in standard form

8. The modulus or absolute value of a complex number $z=a+bi$ is given by:

$$|a+bi|=\sqrt{a^2+b^2}$$

9. The trigonometric or polar form of a complex number $z=a+bi$ is given by:

$$z=r(\cos\theta+i\sin\theta)$$

10. Powers of complex numbers in polar form:

$$z=r(\cos\theta+i\sin\theta)$$
$$z^2=r^2(\cos 2\theta+i\sin 2\theta)$$
$$z^3=r^3(\cos 3\theta+i\sin 3\theta)$$

11. DeMoivre's Theorem

If $z = r(\cos\theta + i\sin\theta)$ is a complex number and n is a positive integer, then
$z^n = r\left[(\cos\theta + i\sin\theta)\right]^n = r^n(\cos n\theta + i\sin n\theta)$

Use DeMoivre's Theorem to find $\left(-1+\sqrt{3i}\right)^{12}$

First convert to polar form:

$$-1+\sqrt{3i} = 2\left(\cos\frac{2\pi}{3} + i\sin\frac{2\pi}{3}\right)$$

Then:

$$= \left[2\left(\cos\left(\frac{2\pi}{3}\right) + i\sin\left(\frac{2\pi}{3}\right)\right)\right]^{12}$$

$$= 2^{12}\left[\left(\cos\left(12\frac{2\pi}{3}\right) + i\sin\left(12\cdot\frac{2\pi}{3}\right)\right)\right]$$

$$= 4096(\cos 8\pi + i\sin 8\pi)$$
$$= 4096$$

12. To convert a complex number into polar or trigonometric form:

1. Calculate the modulus of the complex number, $r = \sqrt{a^2 + b^2}$
2. Calculate $\tan\theta$ where $\tan\theta = \tan\frac{b}{a}$ or $\tan\frac{y}{x}$.
3. Determine θ based upon its tangent value and its quadrant.
4. Substitute r and θ into $z = r(\cos\theta + i\sin\theta)$.