Series Sequences and Summation

Sequences

A sequence is a set of numbers in which each one is related in a definite way to the number that precedes it.

Arithmetic Sequence

An arithmetic sequence is a number series in which each term may be obtained from the preceding one by adding a constant called the common difference. If an arithmetic sequence has first term a_1 and common difference *d*, then a_n the *n*th term of the sequence, is given by:

$$a_n = a_1 + (n-1)d$$

Example:

Find the 25th term of the arithmetic sequence 4 + 7 + 10 + 13 + …
Since $a_1 = 4$, *d* = 3, and *n* = 25, $a_n = 4 + (25-1)3 = 76$

The sum S_n of the first *n* terms of an arithmetic sequence is given by:

$$S_n = \frac{n}{2}\left[2a_1 + \left(n-1\right)d\right]$$

Or equivalently,

$$S_n = \frac{n}{2}\left(a_1 + a_n\right)$$

Example:

Find the sum of the first 6 terms of 4 + 7 + 10 + …

$$S_n = \frac{6}{2}\left[2\left(4\right) + \left(6-1\right)3\right] = 3\left(8+15\right) = 69$$

Geometric Sequence

A geometric sequence is a number series in which each term may be obtained from the preceding one by multiplying by a fixed number called the ratio. If a geometric sequence has a first term a_1 and a common ratio *r*, then the *n*th term of a geometric sequence is given by:

$$a_n = a_1 r^{n-1}$$

Example:

Find the 8th term of the geometric sequence. Since the first term is $a_1 = 3$, the ratio is $r = 2$ and n – 1 = 8 – 1 = 7.

$$a_n = a_1 r^{n-1}$$
$$a_n = 3\left(2^7\right) = 3(128) = 384$$

If a geometric sequence has a first term a_1 and a common ratio r, then the sum S_n of the first n terms is given by:

$$S_n = \frac{a_1\left(r^n - 1\right)}{r - 1},\ r \neq 1$$

Example:

Find the sum of the first 5 terms of 2, 16, 18, …

Since a = 2, n = 5, and r = 3,

$$S_n = \frac{2\left(3^5 - 1\right)}{3 - 1} = \frac{2(243 - 1)}{3 - 1} = \frac{2(242)}{2} = 242$$

The common ratio is determined as follows:

$$r = \frac{a_n}{a_{n-1}}$$

Summation

Summation is the act or process of forming a sum.

Sigma Notation

The Greek letter sigma (Σ) is used for summation notation. The sum of *n* terms $a_1, a_2, a_3, \ldots a_n$ is written as

$$\sum_{i=1}^{n} a_i = a_1 + a_2 + a_3 + \ldots + a_n$$

Where *i* is the index of summation, a_i is the *i*th term of the sum, and the upper and lower bounds of summation are *n* and 1.

For example, the first four terms of the sequence 3, 5, 7, 9,…, 2*k*+1 can be written as follows:

$$\sum_{k=1}^{4} 2k + 1$$

This is read "the sum as *k* goes from 1 to 4 of 2*k*+1.

Summation Rules

1. $\sum_{i=1}^{n} ca_i = c\sum_{i=1}^{n} a_i$
2. $\sum_{i=1}^{n} (a_i \pm b_i) = \sum_{i=1}^{n} a_i \pm \sum_{i=1}^{n} b_i$
3. $\sum_{i=1}^{n} c = c + c + c + \ldots + c$ Equals *c*, *n* times (*cn*), where *c* is a constant.
4. $\sum_{i=1}^{n} i = 1 + 2 + 3 + \ldots + n = \frac{n(n+1)}{2}$
5. $\sum_{i=1}^{n} i^2 = 1^2 + 2^2 + 3^2 + \ldots + n^2 = \frac{n(n+1)(2n+1)}{6}$
6. $\sum_{i=1}^{n} i^3 = 1^3 + 2^3 + 3^3 + \ldots + n^3 = \frac{n^2(n+1)^2}{4}$
7. $\sum_{i=1}^{n} i^4 = \frac{n(n+1)(2n+1)(3n^2+3n-1)}{30}$

Series

An indefinite number of terms succeeding one another, each of which is derived from one or more of the preceding by a fixed law, called the law of the series; as, an arithmetical series; a geometrical series. If the sequence of partial sums converges to a definite value, the series is said to converge. If the sequence of partial sums does not converge to a limit (e.g., it oscillates or approaches $\pm\infty$), the series is said to diverge.

Convergent Series

A convergent series is a series whose sequence of partial sums approaches a limit. The series $\sum_{n=0}^{\infty}\left(\frac{1}{2}\right)^n$ converges since its sequence of partial sums is the sequence $1, 1\frac{1}{2}, 1\frac{3}{4}, 1\frac{7}{8}, \ldots,$ whose limit is 2.

Divergent Series

A divergent series is a series whose sequence of partial sums does not approach a limit. The series $\sum_{n=1}^{\infty}\frac{1}{n}$ does not converge since its sequence of partial sums gets larger without limit.